

On Time-Slotted Communication over Molecular Timing Channels

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ABSTRACT

This work studies time-slotted communication over molecular timing (MT) channels. The transmitter, assumed to be perfectly synchronized in time with the receiver, releases a single information particle in each time slot, where the information is encoded in the time of release. The receiver decodes the transmitted information based on the random time of arrivals of the information particles during a finite-time reception window. The maximum-likelihood (ML) detector is derived and shown to have an exponential computational complexity, thus, rendering it impractical. In addition, two practical detectors are presented: The first is a symbol-by-symbol detector. The second is a sequence detector which is based on the Viterbi algorithm (VA), yet, the VA is used differently than in its common application in ML detection where information is transmitted over linear channels with memory. Numerical simulations indicate that the proposed sequence detection algorithm significantly improves the performance compared to the symbol-by-symbol detector. Furthermore, for a short number of transmitted symbols it closely approaches the highly complicated ML detector.

1. INTRODUCTION

Molecular communication (MC) is a new communication paradigm in which nano-scale devices communicate with each other by exchanging small *information particles* [1]. Several methods are used in MC systems to encode the transmitted information: in the type of the released particles, in their concentration, in their number, or in their time of release [2, 3]. There are also several mechanisms for transporting the information particles from the transmitter to the receiver: diffusion, active transport, bacteria, and flow. In this work, we study receiver design for MC systems in which the information is encoded in the *time of release of the information particles*, see, for example, [4, 5]. This model is also referred to as the molecular timing (MT) channel. We use the common assumption, which is accurate for many sensors, that each particle which arrives at the receiver during its observation window is absorbed and removed from the environment. Thus, the random delay until a particle arrives at the receiver can be represented as an additive noise term. Moreover, particles which do not arrive within the receiver's observation window are assumed to be destroyed. For an unbounded observation window,

and when the transportation mechanism is diffusion *without flow*, this additive noise follows the Lévy distribution [6], whereas with flow the noise follows an inverse Gaussian (IG) distribution [7].¹

Traditional electromagnetic (EM) communication and MC share several similarities which motivates using tools and algorithms common in EM communication systems in designing MC receivers. On-off transmission based on diffusion of information particles was studied in [8, 9]. In this setup the receiver recovers the transmitted information from the measured concentration of information particles. The work [8] modeled this channel as a linear channel with finite memory and additive Gaussian noise, and derived the optimal, i.e., maximum likelihood (ML), sequence detector using the Viterbi Algorithm (VA) [10]. A similar setup was studied in [11], which derived a technique for inter-symbol interference (ISI) mitigation along with a reduced-state ML sequence detection algorithm. On-off transmission over a diffusive MC channel with flow was studied in [12], which proposed an ML sequence detection algorithm, and designed a family of weighted sums detectors.

The above works use a linear channel model with additive (and in some cases Gaussian) noise. Yet, MT channels are *not linear* and the additive noise is not Gaussian. Moreover, in MT channels the symbol duration is a random variable (RV), and therefore, information particles may arrive out-of-order. Decoding particles in the correct order is thus a big challenge for the detector, in particular when the transmitted information particles are indistinguishable [13].

Main Contributions: Previous studies on MT channels either considered a memoryless channel, see [3] and references therein, or focused on the information theoretic aspects of the problem [2, 13]. In this work, we study a more practical setting of time-slotted communication in which the receiver has a *bounded observation window*. This implies that *not only do the transmitted particles arrive out-of-order but, in addition, some of them do not arrive at the receiver within its observation time*. We derive the ML detector for this problem and show that it has an exponential complexity. As the channel is not linear, and due to the lack of ordering, this ML detector cannot be efficiently implemented using an algorithm with polynomial complexity. Therefore, we first derive a simple symbol-by-symbol detector, which extends the detector presented in [14]. We next develop a sequence detection algorithm that is based on the VA. However, in contrast with the traditional VA algorithm in EM communication, our detector uses the VA in the context of hidden

¹Note that in [7] a single-dimension environment is studied.

Markov models. Via numerical simulations we show that the proposed sequence detection algorithm significantly improves the performance compared to the symbol-by-symbol detector, and for a short number of transmitted symbols it closely approaches the highly complicated ML detector.

The rest of this paper is organized as follows. The problem formulation and some preliminaries are presented in Section 2. Optimal detection in the presence of ISI is studied in Section 3. The symbol-by-symbol detector is presented in Section 4, while the sequence detector is derived in Section 5. Numerical results are presented in Section 6 and concluding remarks are provided in Section 7.

Notation: We denote the set of real numbers with \mathfrak{R} , the set of positive real numbers with \mathfrak{R}^+ , and the set of integers with \mathcal{N} . Other than these sets, we denote sets with calligraphic letters, e.g., \mathcal{V} . We denote RVs with upper case letters, e.g., X , and their realizations with lower case letters, e.g., x . We use $f_Y(y)$ to denote the probability density function (PDF) of a continuous RV Y on \mathfrak{R} , $f_{Y|X}(y|x)$ to denote the conditional PDF of Y given X , and $F_{Y|X}(y|x)$ to denote the conditional cumulative distribution function (CDF). Finally, we use $\text{erfc}(\cdot)$ to denote the complementary error function given by $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$, and $\log(\cdot)$ to denote the natural logarithm.

2. PRELIMINARIES

2.1 System Model

We assume that the information particles themselves are *identical and indistinguishable* at the receiver. Therefore, the receiver can only use the time of arrival to decode the intended message. The information particles propagate from the transmitter to the receiver through some random propagation mechanism (e.g. diffusion). We make the following assumptions about the system:

- A1)** The transmitter perfectly controls the release time of each information particle, and the receiver perfectly measures the arrival times of the information particles. Furthermore, the transmitter and the receiver are perfectly synchronized in time.
- A2)** An information particle which arrives at the receiver is absorbed and removed from the propagation medium.
- A3)** All information particles propagate independently of each other, and their trajectories are random according to an i.i.d. random process.²

We note that these assumptions are consistent with those made in all previous works, in order to make the models tractable (see [3] and references therein).

Let \mathcal{X} be a finite set of constellation points on the real line: $\mathcal{X} \triangleq \{\xi_0, \xi_1, \dots, \xi_{L-1}\}$, $0 \leq \xi_0 \leq \dots \leq \xi_{L-1} \leq T_s$, where $T_s < \infty$ denotes the symbol duration. The k^{th} transmission takes place at time $(k-1)T_s + X_k$, $X_k \in \mathcal{X}$, $k = 1, 2, \dots, K$. At this time, a *single* information particle is released into the medium by the transmitter. The transmitted information is encoded in the sequence $\{(k-1)T_s + X_k\}_{k=1}^K$, which is assumed to be independent of the random propagation time of *each* of the information particles. Let Y_k denote the time of arrival of the information particle released at time $(k-$

²This is a reasonable assumption for many different propagation schemes in molecular communication such as diffusion in dilute solutions, i.e., when the number of particles released is much smaller than the number of molecules of the solutions.

$1)T_s + X_k$. It follows that $Y_k > X_k$. Thus, we obtain the following additive noise channel model:

$$Y_k = (k-1)T_s + X_k + Z_k, \quad k = 1, 2, \dots, K, \quad (1)$$

where Z_k , is a random noise term representing the propagation time of the particle transmitted at the k^{th} time slot.³

In the rest of this work we restrict our attention to the case of binary modulations and set $\mathcal{X} = \{0, \Delta\}$. Let $\{S_k\}_{k=1}^K$, $S_k \in \{0, 1\}$, denote a sequence of independent and equiprobable bits to be sent over the channel to the receiver, where the k^{th} bit is sent in the k^{th} slot. Without loss of generality assume that the bit $b \in \{0, 1\}$ is mapped to the symbol ξ_b . Let \hat{S}_k denote the estimate of S_k at the receiver, and define $P_{\varepsilon,k} \triangleq \Pr\{\hat{S}_k \neq S_k\}$. Our objective is to design receivers that minimize the following average probability of error:

$$P_\varepsilon = \frac{1}{K} \sum_{k=1}^K P_{\varepsilon,k}. \quad (2)$$

REMARK 1. *In contrast to EM communication in which $P_{\varepsilon,k}$ is roughly the same for all k ,⁴ in our setup the probability of error of each symbol is different. More precisely, for values of T_s which are close to Δ , each symbol experiences ISI from all previous transmitted symbols, implying that the probability of error is different for every k . For instance, for $K = 2$, in the first slot only the first particle may arrive, while in the second slot both the first and the second particles may arrive. On the other hand, when $T_s \gg \Delta$ then there is almost no ISI, and all symbols have roughly the same $P_{\varepsilon,k}$.*

We emphasize that the above description of communication over MT channels is fairly general and can be applied to different propagation mechanisms as long as Assumptions **A1)**–**A3)** are not violated. In the rest of the paper we focus on *diffusion-based propagation without flow* which is governed by the Lévy distribution.

2.2 The Diffusion-Based MT Channel

In diffusion-based propagation, the released particles follow a random Brownian path from the transmitter to the receiver. In this case, to specify the random additive noise term Z_k in (1), we define a Lévy-distributed RV as follows:

DEFINITION 1. *Let Z be Lévy-distributed with location parameter μ and scale parameter c [15]. Then, its PDF is given by:*

$$f_Z(z) = \begin{cases} \sqrt{\frac{c}{2\pi(z-\mu)^3}} \exp\left(-\frac{c}{2(z-\mu)}\right), & z > \mu \\ 0, & z \leq \mu \end{cases}, \quad (3)$$

and its CDF is given by:

$$F_Z(z) = \begin{cases} \text{erfc}\left(\sqrt{\frac{c}{2(z-\mu)}}\right), & z > \mu \\ 0, & z \leq \mu \end{cases}. \quad (4)$$

Let d denote the distance between the transmitter and the receiver, and D denote the diffusion coefficient of the information particles in the propagation medium. Following along the lines of the derivations in [7, Sec. II], and using the results of [16, Sec. 2.6.A], it can be shown that for the 1-dimensional pure diffusion, the propagation time of each of the information particles follows a Lévy distribution, denoted in this work by $\sim \mathcal{L}(\mu, c)$ with $c = \frac{d^2}{2D}$ and $\mu = 0$. Thus, $Z_k \sim \mathcal{L}(0, c)$, $k = 1, 2, \dots, K$.

³Note that Assumption **A3)** implies that all the RVs Z_k are independent.

⁴Here we neglect initialization and termination effects of communication in the presence of ISI.

REMARK 2. The work [17] showed that a scaled Lévy distribution can also model the first arrival time in the case of an infinite, three-dimensional homogeneous medium without flow. Hence, our results can be extended to 3-D space by simply introducing a scalar factor.

2.3 Optimal Detection When $T_s \gg \Delta$

Before studying detection for arbitrary values of T_s , we first briefly discuss the case of $T_s \gg \Delta$, namely, the channel is (approximately) memoryless and optimal detection can be applied for each symbol separately. Clearly, the performance in this case can serve as a lower bound on the performance in the case of arbitrary T_s . The optimal decision rule, assuming a memoryless channel, was presented in [14, Prop. 1], repeated here for ease of reference:

PROPOSITION 1. Let $y = y_k - (k - 1)T_s$. The decision rule which minimizes the probability of error for $T_s \gg \Delta$, is given by:

$$\hat{X}_{No-ISI}(y) = \begin{cases} 0, & y < \theta \\ \Delta, & y \geq \theta, \end{cases} \quad (5)$$

where θ is the unique solution, in the interval $[\Delta, \Delta + \frac{c}{3}]$, of the following equation in y :

$$y(y - \Delta) \log \left(\frac{y_1}{y_1 - \Delta} \right) = \frac{c\Delta}{3}, \quad y > \Delta > 0. \quad (6)$$

Furthermore, the probability of error of this decision rule is given by:

$$P_\varepsilon = 0.5 \left(1 - \operatorname{erfc} \left(\sqrt{\frac{c}{2\theta}} \right) + \operatorname{erfc} \left(\sqrt{\frac{c}{2(\theta - \Delta)}} \right) \right). \quad (7)$$

Next, we study detection for arbitrary values of $T_s > \Delta$, namely, when the channel has memory.

3. ML DETECTION FOR ARBITRARY T_s

We begin this section with the observation that ISI in MT channels is fundamentally different from ISI in traditional EM communication [2, 13]. More specifically, in EM communication the channel is commonly assumed to be linear: $v_k = h_0\zeta_k + \sum_{j=1}^L h_j\zeta_{k-j} + w_k$, where $\{\zeta_k\}$ are the channel inputs, $\{h_j\}_{j=0}^L$ are the (usually known) channel response coefficients, w_k is an additive noise independent of the channel coefficients and of the transmitted signal, and v_k is the channel output. Thus, a symbol transmitted at time index k systematically affects the current and future L output symbols via the channel response $\{h_j\}_{j=0}^L$.

On the other hand, in MT channels the ISI is random, namely, there is a non-negligible probability that a particle that was transmitted at time slot $n < k$ will arrive at time slot k . Furthermore, in MT channels an information particle arrives only once at the receiver, this is in contrast to EM communications in which ζ_k is assumed to be observed at the receiver both at time instance k and $k + l$, $l < L$. Finally, in MT channels, due to the random delay, the channel outputs are not ordered, and some time slots may even have no arrivals.

REMARK 3. Part of the above observations were stated in [18, Remark 1]. Yet, in [18] the authors considered on-off concentration modulation and tackled the above challenge by analyzing the expected channel output. In diffusion-based MT channels the mean and variance of the channel output are not defined (since the Lévy noise has no mean and variance). Thus, this approach is not applicable.

Let $\{t_1 \leq t_2 \leq \dots \leq t_n\}$, $1 \leq n \leq K$ be the sequence of arrival times at the receiver in the considered K time slots. Thus, $t_n \leq KT_s$. Note that $t_j \in \{y_1, y_2, \dots, y_K\}$, yet, due to the fact that the particles are indistinguishable, the receiver does not know the mapping $\{t_j\}_{j=1}^n \mapsto \{y_k\}_{k=1}^K$. Further note that it is not guaranteed that all particles transmitted in these time slots indeed arrive, thus $n \leq K$.

Let $\operatorname{perm}(\mathbb{M})$ denote the permanent of a square matrix \mathbb{M} , see [19], and let \mathbf{x} denote a short-hand notation for $\{x_k\}_{k=1}^K$. Further define $f_k(y, \mathbf{x}) \triangleq f_{Y|X}(y|x_k)$, where $f_{Y|X}(y|x_k) = f_Z(y|\mu = x_k)$. We define $F_k(y, \mathbf{x})$ in a similar manner. The detector that minimizes the probability of error is stated in the following theorem:

THEOREM 1. The following detector minimizes the probability of error in detecting \mathbf{X} based on $\{t_1, t_2, \dots, t_n\}$:

$$\hat{\mathbf{X}}_{ML} = \operatorname{argmax}_{\mathbf{x}} \operatorname{perm}(\mathbb{M}(\mathbf{x})), \quad (8)$$

where $\mathbb{M}(\mathbf{x})$ is a $K \times K$ matrix given by:

$$\mathbb{M}(\mathbf{x}) = \begin{bmatrix} f_1(t_1, \mathbf{x}) & \dots & f_K(t_1, \mathbf{x}) \\ f_1(t_2, \mathbf{x}) & \dots & f_K(t_2, \mathbf{x}) \\ \vdots & & \vdots \\ f_1(t_n, \mathbf{x}) & \dots & f_K(t_n, \mathbf{x}) \\ 1 - F_1(t_n, \mathbf{x}) & \dots & 1 - F_K(t_n, \mathbf{x}) \\ \vdots & & \vdots \\ 1 - F_1(t_n, \mathbf{x}) & \dots & 1 - F_K(t_n, \mathbf{x}) \end{bmatrix},$$

with the last $K - n$ rows of $\mathbb{M}(\mathbf{x})$ identical and equal to:

$$[1 - F_1(t_n, \mathbf{x}) \quad \dots \quad 1 - F_K(t_n, \mathbf{x})].$$

REMARK 4. While the works [2, 13] also considered optimal detection when order is not preserved, they assumed that all the particles arrive and expressed the joint density of the sequence $\{t_j\}_{j=1}^K$ in terms of a folded density of $\{y_k\}_{k=1}^K$, see [13, Eq. (4)].

PROOF OF THEOREM 1. As all the symbols are equiprobable, the detector that minimizes the probability of error is the ML detector, given by:

$$\hat{\mathbf{X}}_{ML} = \operatorname{argmax}_{\mathbf{x}} f_{\{T_j\}_{j=1}^n | \{X_k\}_{k=1}^K}(\{t_j\}_{j=1}^n | \{x_k\}_{k=1}^K).$$

Note that $\{t_j\}_{j=1}^n$ are the first n arrivals, or equivalently, the smallest n values in the sequence $\{y_k\}_{k=1}^K$. Further note that given $\{x_k\}_{k=1}^K$, the RVs $\{Y_k\}_{k=1}^K$ are independent but non-identically distributed, since each Y_k has a different offset parameter. Note that given x_k , the PDF of Y_k is $f_k(y, x_k)$, and its CDF is $F_k(y, x_k)$. Thus, using results from order-statistics theory [20, pg. 309],⁵ we have:

$$f_{\{T_j\}_{j=1}^n | \{X_k\}_{k=1}^K}(\{t_j\}_{j=1}^n | \{x_k\}_{k=1}^K) = \frac{1}{(K-n)!} \operatorname{perm}(\mathbb{M}(\mathbf{x})).$$

This concludes the proof. \square

REMARK 5. In traditional EM communication the ML detector can be implemented using the VA which requires a polynomial computational complexity in K . On the other hand, the computational complexity of the ML detector in (8), which uses an exhaustive search, is $\mathcal{O}(2^{2K} K^2)$. This follows as the best known algorithm for calculating the permanent is by Ryser from 1963, with the complexity of $\mathcal{O}(2^K K^2)$,

⁵Note that we are interested in the density of the smallest n RVs out of k independent RVs.

see [21, Ch. 1.3], while this permanent should be evaluated for each possible sequence $\{x_k\}_{k=1}^K$. Since only calculating the permanent requires exponential complexity, we conjecture that the ML detector in (8) cannot be implemented efficiently, i.e., in polynomial time.

As the ML detector cannot be implemented even for moderate values of K , we next consider sub-optimal detectors. In the next section we consider a simple symbol-by-symbol detector which is asymptotically optimal, i.e., for $T_s \gg \Delta$.

4. SYMBOL-BY-SYMBOL DETECTION

We first note that the detector in (5) should be adapted to the setting in which ISI is present, since there might be more than a single arrival within the slot boundaries. This is again fundamentally different than in EM communication where the optimal symbol-by-symbol detector can simply be used in the presence of ISI, at the cost of higher probability of error. We propose the following adaptation to the detector in (5):

A Symbol-by-symbol decision rule: Let α_k denote the number of arrivals in the time interval $[(k-1)T_s, (k-1)T_s + \theta)$, $k = 1, 2, \dots, K$, and β_k denote the number of arrivals in the interval $[(k-1)T_s + \theta, kT_s)$. The detector applies the following decision rule:⁶

$$\hat{S}_k(\alpha_k, \beta_k) = \begin{cases} 0, & \alpha_k > 0 \text{ and } \beta_k = 0 \\ 1, & \alpha_k = 0 \\ q_k, & \alpha_k > 0 \text{ and } \beta_k > 0, \end{cases} \quad (9)$$

where $q_k \in \{0, 1\}$ is a Bernoulli RV with $\Pr\{q_k = 0\} = 0.5$, i.i.d. over time.

The first two events on the right hand side (RHS) of (9) are an extension of (5) to the case of multiple arrivals at the same time slot. On the other hand, for the third event we note that, as we cannot distinguish between different information particles, it is not clear which decision should be taken. Therefore, we simply toss a coin via the RV q_k . Numerical simulations indicate that other policies yield roughly the same probability of error.⁷

In [22, Theorem 2] we derive the exact probability of error of this adapted detector. We omit this expression here due to space limitations. In [22, Corollary 1] we also show that for $T_s \gg \Delta$ the performance of the adapted detector approaches (7). An intuitive explanation for this result is the observation that when $T_s \gg \Delta$, then the probability that a particle will not arrive in its time slot is negligible, and (9) specializes to (5).

Next, we present our proposed sequence detection algorithm.

5. SEQUENCE DETECTION ALGORITHM

We begin this section with the observation that for small values of T_s , or for heavy-tailed noise distributions like the Lévy distribution, the channel (1) is reminiscent of an infinite impulse response (IIR) channel in EM communication, in the sense that the current channel input depends on *all* previous transmitted symbols. Yet, in contrast to the commonly linear model, the channel (1) is not linear. Moreover, in some of the slots there are no arrivals, while the total

⁶Note that this detector is not necessarily the optimal symbol-by-symbol detector in the presence of ISI, yet, asymptotically, i.e., for $T_s \gg \infty$, it achieves the performance of the detector in (5).

⁷For instance, majority rule together with tossing a coin when $\alpha_k = \beta_k$.

number of arrivals in the receiver observation window can be smaller than the number of transmissions. Therefore, applying traditional sequence detection algorithms is far from straight-forward. To tackle the challenge of the lack of measurements in arbitrary slots, we use a simplified discrete (in amplitude) channel model which counts the number of arrivals at each slot. Thus, we obtain valid measurements even for slots with no arrivals. Then, we apply sequence detection based on the simplified model using a modified VA. Clearly, as we use a simplified channel model, the resulting detector is sub-optimal. Yet, simulation results show that it can achieve performance close to the ML detector.

In its most general form, the VA solves a MAP estimation problem of the state sequence of a *finite-state* discrete-time Markov process observed in memoryless noise [10]. However, the channel in our problem may have *infinite* memory which implies an *infinite state space*. This is tackled by truncating the channel response. More precisely, by noting that the density of the noise decays with time, it follows that for transmission time and observation time far enough apart, one can approximate this density by zero, thus, truncating the channel and resulting in a channel with finite memory. Yet, it should be noted that in contrast to EM communication, in our setting such truncation requires separate treatment for cases in which the observed channel output is not a valid output of the assumed model. We denote this event as an *out of model event*.

5.1 From Arrival Times to Arrival Counts

To overcome the lack of measurements in arbitrary slots, and to partially cope with the lack of ordering, we calculate the sequence $\{V_k\}_{k=1}^K$, $V_k = (\alpha_k, \beta_k)$, as defined in Section 4, from the sequence $\{t_j\}_{j=1}^n$, and use $\{V_k\}_{k=1}^K$ as input to our algorithm. Note that with this transformation we have a valid measurement for every time slot. It should be noted that the proposed detection algorithm can be used with any partition of the slot. We choose the above partition since it minimizes the probability of error in a single symbol transmission and it achieves the optimal detection performance for $T_s \gg \Delta$.

5.2 Truncating the Channel

To obtain a finite state model we truncate the channel, namely, we assume that in the k^{th} time slot one can observe arrivals of information particles transmitted in time slots $\{k-l\}_{l=0}^L$, thus, obtaining a channel of memory L . Clearly, larger L captures the channel statistics more accurately at the cost of exponentially increased computational complexity. We propose to choose L as a given fraction of the worst case probability of arrival of past information particles. Let p_k^{ISI} denote the probability that *any past* transmitted information particle arrives at time slot k . We have the following proposition:

PROPOSITION 2. *The probability of arrival of past information particles at time slot k increases with k and is given by $p_k^{\text{ISI}} = \sum_{i=1}^{k-1} p_{k,i}^{\text{ISI}}$, where $p_{k,i}^{\text{ISI}}$ is defined in (10) at the top of the next page.*

PROOF. The proof is provided in [22]. \square

The above proposition implies that the *last slot experiences the largest ISI*. Thus, L can be chosen to cover a given fraction of p_K^{ISI} .

$$p_{k,i}^{\text{ISI}} = 0.5 \left\{ \operatorname{erfc} \left(\sqrt{\frac{c}{2(k-i+1)T_s}} \right) - \operatorname{erfc} \left(\sqrt{\frac{c}{2(k-i)T_s}} \right) + \operatorname{erfc} \left(\sqrt{\frac{c}{2((k-i+1)T_s - \Delta)}} \right) - \operatorname{erfc} \left(\sqrt{\frac{c}{2((k-i)T_s - \Delta)}} \right) \right\}. \quad (10)$$

5.3 The Markov Structure

For a given L , we *assume* a finite-memory channel, i.e., $V_k = (\alpha_k, \beta_k)$ depends on the transmissions at time slots $k-L, k-L+1, \dots, k$. Thus, we write:

$$\Pr\{\{V_k\}_{k=1}^K | \{X_k\}_{k=1}^K\} = \prod_{j=1}^L \Pr\{V_j | \{X_i\}_{i=1}^j, \{V_i\}_{i=1}^{j-1}\} \\ \times \prod_{k=L+1}^K \Pr\{V_k | \{X_i\}_{i=k-L}^k, \{V_i\}_{i=1}^{k-1}\}. \quad (11)$$

Note that in (11), V_k depends on all the previous V 's, thus it does not represent a process with finite memory. This follows from the fact that each particle can arrive at the receiver only once. Motivated by the case of $T_s \gg \Delta$, or by the large Δ regime,⁸ in which the dependence between V_k and $\{V_j\}_{j=1}^{k-1}$ is very weak, we apply the approximation $\Pr\{V_k | \{X_i\}_{i=k-L}^k, \{V_i\}_{i=1}^{k-1}\} \approx \Pr\{V_k | \{X_i\}_{i=k-L}^k\}$, which results in a finite memory. In Section 6 we show that even with this approximation, in some settings the performance of the proposed sequence detector approaches the performance of the ML detector. With this approximation in hand, we now propose the following sequence detector:

$$\hat{\mathbf{X}}_{\text{SD}} = \operatorname{argmax}_{\{x_k\}_{k=1}^K} \sum_{j=1}^L \log \left(\Pr\{V_j | \{x_i\}_{i=1}^j\} \right) \\ + \sum_{k=L+1}^K \log \left(\Pr\{V_k | \{x_i\}_{i=k-L}^k\} \right). \quad (12)$$

The first summation on the RHS of (12) constitute L steps of initializations, while the second summation can be efficiently calculated using the VA. Next, we briefly elaborate on the initialization of the VA, the calculation of the steady-state branch metric, and the traceback.

5.4 Branch Metric, Initialization, and Traceback

Branch Metric: Equation (12) implies that the trellis consists of 2^L states, characterized by $\{x_i\}_{i=k-L}^{k-1}$, and the transition metric is given by $\log(\Pr\{V_k = v | \{x_i\}_{i=k-L}^k\})$. Note that since we only *assume* a truncated channel model, v in this metric *may not* belong to the set of possible output pairs defined by the assumed model. More precisely, the assumption of finite memory of order L implies that $V_k \leq L+1, k > L$. On the other hand, the *measured* V_k is generated by the original channel, and therefore $V_k \leq k+1$. To tackle this mismatch we first define the set \mathcal{V}_{tm} which contains all the possible V 's in our truncated channel model $\mathcal{V}_{\text{tm}} \triangleq \{(\alpha, \beta) : \alpha + \beta \leq L+1, \alpha \in \mathfrak{R}, \beta \in \mathfrak{R}\}$. Next, we define the set $\mathcal{V}_{\text{res}} = \{(\alpha, \beta) : L+1 < \alpha + \beta \leq k+1, \alpha \in \mathfrak{R}, \beta \in \mathfrak{R}\}$, namely, \mathcal{V}_{res} is the set of all V 's which cannot be generated by our model. The branch metric is now given by:

$$\text{TM}(v, \{x_i\}_{i=k-L}^k) \\ = \begin{cases} \log(\Pr\{v | \{x_i\}_{i=k-L}^k\}), & v \in \mathcal{V}_{\text{tm}}, \\ \log \left(1 - \sum_{(\tilde{v} \in \mathcal{V}_{\text{tm}})} \Pr\{\tilde{v} | \{x_i\}_{i=k-L}^k\} \right), & v \in \mathcal{V}_{\text{res}}. \end{cases} \quad (13)$$

⁸These are reminiscent of the high SNR regime in EM communication.

REMARK 6. The terms $\log(\Pr\{v | \{x_i\}_{i=k-L}^k\})$ are calculated based on the finite memory model, namely, without accounting for the tails of the noise density. Therefore, as the support of the Lévy distribution is \mathfrak{R}^+ , the probability of the out of model event, given by $1 - \sum_{(\tilde{v} \in \mathcal{V}_{\text{tm}})} \Pr\{\tilde{v} | \{x_i\}_{i=k-L}^k\}$, is larger than zero.

Initialization and traceback: In the initialization phase we use the metric $\log(\Pr\{V_j = v | \{x_i\}_{i=1}^j\})$. Here, as $j < L$, the out of model event is not required. Finally, as there is no termination transmission, we trace back starting from the state with the maximal accumulated metric.

REMARK 7. In [22, Section 12] we consider the case of $L = 1$, detail the sets \mathcal{V}_{tm} and \mathcal{V}_{res} , and describe how the respective transition probabilities are calculated.

6. NUMERICAL RESULTS

Next, we demonstrate our results via numerical simulations. To simplify the presentation we assume that the slot length scales proportionally to Δ , and present the average probability of error as a function of Δ for a given value of c . The parameter c is known to be the dispersion of the noise [14], which is reminiscent of the standard deviation in the case of the Gaussian distribution. This implies that larger values of c correspond to more ISI.

In Fig. 1 we present P_ϵ versus Δ for $c = 2$ and $T_s = 1.5\Delta$. Here, as the ML detector has exponential complexity, we consider a short block of length $K = 6$. The sequence detector is implemented with $L = 2$, while for each Δ point 10^4 trials were carried out. It can be observed that the sequence detector proposed in Section 5 achieves almost the same probability of error as the ML detector, yet, it entails only a polynomial complexity. It can further be observed, as expected, that both detectors are significantly outperformed by the no-ISI detector; this is due to the severe ISI which results in non-ordered arrivals, or even arrival of only part of the information particles. Fig. 1 also indicates the large gains of the sequence detector over the symbol-by-symbol detector. For example, to achieve $P_\epsilon = 0.1$ the sequence detector requires $\Delta = 49[\text{s}]$, which reflects a gain of 25% compared to the symbol-by-symbol detector.

Fig. 2 depicts P_ϵ versus Δ for $c = 0.1$ and $T_s = 15\Delta$. Here $K = 100$, and the ML detector is not presented due to its exponential computational complexity. It can be observed that the performance of the sequence detector is very close to the optimal detector when no ISI is present. This is due to the relatively large spacing between transmissions. On the other hand, for $T_s = 15\Delta$, one can again observe a significant gain of the sequence detector compared to the symbol-by-symbol detector.

7. CONCLUSIONS

In this work we have studied time-slotted communication over diffusion-based MT channels, where we focused on a practical setting in which the receiver has a bounded observation window. Thus, not only do the transmitted particles arrive out-of-order but, in addition, some of them do not arrive at the receiver within the observation time. We derived three detectors for this communication system. The first is

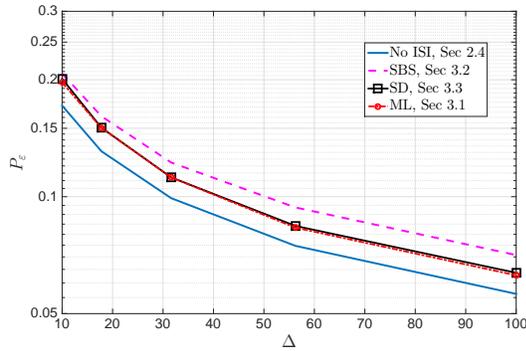


Figure 1: P_e vs. Δ for $c = 2$ and $T_s = 1.5\Delta$. The sequence detection is implemented assuming $L = 2$.

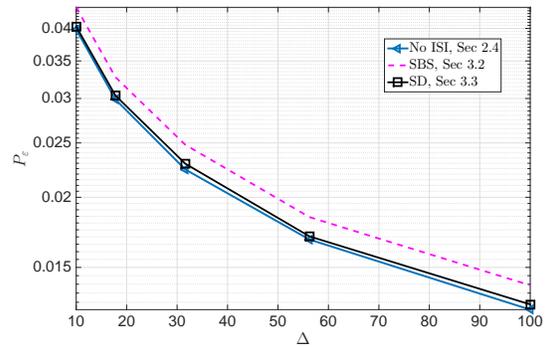


Figure 2: P_e vs. Δ for $c = 0.1$ and $T_s = 15\Delta$. The sequence detection is implemented assuming $L = 2$.

the ML detector which achieves optimal performance at the cost of exponential complexity. The second is a symbol-by-symbol detector. The third is a sequence detector which is based on the VA. However, the VA is used differently than in its common application of ML detection where information is transmitted over linear channels with ISI. Numerical simulations indicate that the proposed sequence detector, which has polynomial complexity, significantly improves the performance compared to the symbol-by-symbol detector. Moreover, for a short number of transmitted symbols its performance is very close to the performance of the exponentially-complex ML detector.

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